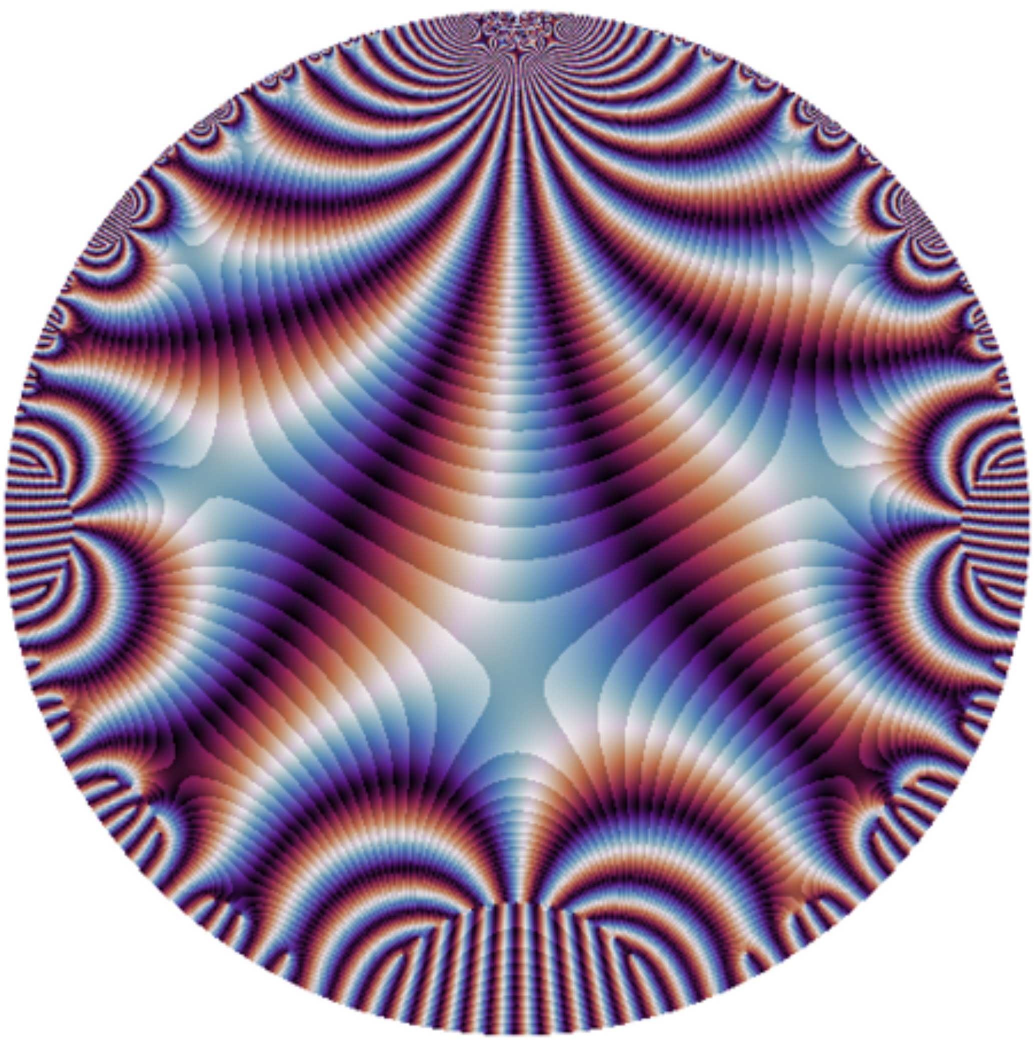
Holographic CFT$_2$: Modular Forms Applied to AdS3 Quantum Gravity



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1 Introduction

Motivation and background. Conformal field theory (CFT) is a powerful framework for studying the universal properties of quantum field theories at critical points. In two dimensions, CFTs have a rich mathematical structure applied in various areas of physics ranging from statistical mechanics and string theory. Not to mention the AdS3/CFT_2 correspondence allowing CFT descriptions of black holes.

This essay will introduce the fundamental concepts of 2d CFTs, focusing on the torus and its modular properties. Using this, we will derive the Cardy formula, which relates the density of states in a CFT to fundamental properties of the theory; the central charge and the conformal dimensions. The Cardy formula has found important applications in black hole physics, e.g. computing the entropy of the BTZ black hole. This essay will focus on the Hawking-Page transition of a black hole, a transition between a thermal state and a non-thermal state of a black hole, which can be described purely from a conformal field theory perspective. The essay's central theme will be the modular invariance of the partition function. Indeed, we will proceed only from a CFT perspective, highlighting the assumptions we make on the way, and see what universal properties we get.

Main goal.

Understand the modular invariant 2d CFT on a torus, its algebra and what assumptions are needed in order to derive universal properties of the theory.

Outline. The essay is organised as follows. In Section 2, we will introduce the two-dimensional conformal field theory, starting from the beginning and building our way up to the torus partition function. In Section 3, we will use the torus partition function to derive the asymptotic formula for density of states; the Cardy formula. In Section 4 we will look at holographic CFTs and study the Hawking-Page transitions. We summarise and discuss the main results in Section 5.

2 The 2d Conformal Field Theory

In this chapter we will give a brief recap of the two-dimensional conformal field theory, the Virasoro algebra, 2d CFT Hilbert space, and different types of CFTs. In the end we will introduce the modular invariant partition function of the 2d CFT on a torus, which will be useful for following chapters. This chapter is aimed towards a student familiar with quantum field theory, general relativity and the most essential concepts of CFT. The material forming this chapter is based on [4, 5, 1].

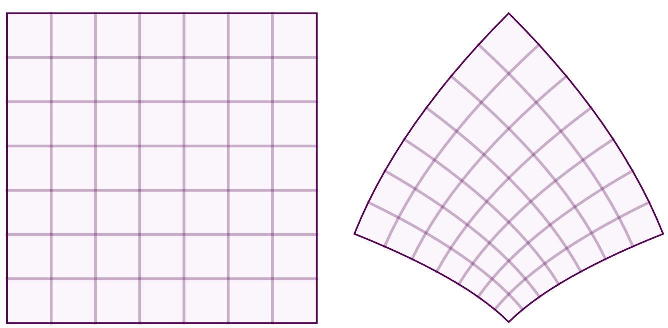
2.1 Conformal Transformations

Let us begin with the key definition of the field theories we will consider in this essay.

Definition 2.1. Given a metric $g_{\mu\nu}$ (in this essay we will consider flat Minkowski space), a *conformal transformation* of the coordinates, is an invertible mapping $\boldsymbol{x} \rightarrow \boldsymbol{x}'$, which leaves the metric tensor invariant up to an overall rescaling:

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) = \Lambda(x) g_{\mu\nu}(x). \tag{1}$$

The *conformal group* is the subgroup of these coordinate transformations. For instance, we can immediately observe that the Poincaré group is a subgroup, since that corresponds to when the metric is invariant; $\Lambda(x) = 1$. This transformation *preserves the angle* between two vectors u, v: $u \cdot v / \left(u^2 v^2\right)^{1/2}$ (where $u \cdot v = g_{\mu\nu} u^\mu v^\nu$). See Fig 1 for an example of a conformal mapping of Cartesian coordinates. A *conformal field theory* is a field theory which is invariant under these transformations, e.g. we can look at it like the physics of such theory looks the same at all length scales.



(a) **(b)**

Figure 1: Conformal mapping transforming Cartesian coordinates (a) into curvilinear coordinates (b), preserving the 90° angles.

Recalling complex analysis, the only angle preserving transformations of the extended complex plane, or Riemann sphere, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are the group of *Möbius transformations*

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d, z \in \mathbb{C},$$

where for invertibility we require $ad - bc \neq 0$. If this is the case, we can then scale the constants a, b, c, d such that $ad - bc = 1$. The group of these transformations is also called the *global conformal group* $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2 = \mathrm{PSL}(2, \mathbb{C})$. The quotient by $\mathbf{Z}_2$ is due to the fact that the expression above is invariant under $(a, b, c, d) \rightarrow (-a, -b, -c, -d)$. The group generators are translations, rotations, dilations, and inversions:

$$f_{\text{tr}}(z) = z + a, \quad f_{\text{rot}}(z) = e^{i\theta} z, \quad f_{\text{dil}}(z) = cz, \quad f_{\text{inv}}(z) = \frac{1}{z}.$$

We note that the global conformal group extend the group of rigid transformations (translations and rotations) by adding the scale transformations and the inverse maps that turns the complex plane "inside out".

Every holomorphic function $z \mapsto f(z)$ is angle-preserving. But if f is not the above Möbius function, then it cannot be one-to-one and must have singularities. From this we can draw the conclusion that it is only locally conformal.

Locally, we can express every meromorphic function with a Laurent expansion $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, so it is linearly generated by the z^n. Accordingly, the infinite-dimensional algebra of local conformal transformations, the familiar *Witt algebra* $\mathfrak{witt}$, is infinitesimally generated by similar objects. Its generators are denoted by ℓ_n, satisfy the following commutation relations:

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z} \quad \Longrightarrow \quad [\ell_n, \ell_m] = (n-m)\ell_{n+m}, \quad n \in \mathbb{Z}.$$

We can relate this to the subalgebra $\mathrm{PSL}(2,\mathbb{C})$ of *global* conformal transformations is generated by the three $\ell_{-1}, \ell_0, \ell_1$. The rest of the Witt generators describe *local* conformal transformations.

2.2 The Virasoro Algebra

In order to get the symmetry algebra of the CFT, which acts on its Hilbert space, we will need to complexify and centrally extend the Witt algebra. We will introduce the complex basis $\left(\ell_n, \bar{\ell}_m\right)$ that consists of two identical and mutually commuting copies of the Witt algebra. The ℓ_n generate the "left-moving" or "holomorphic" conformal transformations, and the $\bar{\ell}_m$ generate the "right-moving" or "anti-holomorphic" ones.

The unique central extension of the Witt algebra the *Virasoro algebra*, with its generators L_n, satisfying:

$$\boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{n,-m}} \quad c \in \mathbb{C},$$

with an analogue expression for $\bar{L}_n$ with $\bar{c}$. Of course, $\left[L_n, \tilde{L}_m\right] = 0$. We have now encountered an important characterisation of the CFT; the *central charge* c, satisfying $[c, L_n] = 0$. Roughly speaking, it is a measure of the number of degrees of freedom in the theory. We also note that setting $c = 0$ gives the Witt algebra, so the extension does not does not affect the generators of the global conformal group.

In summary, the symmetry algebra of a 2d CFT, $\mathfrak{witt} \times \overline{\mathfrak{vir}}$, is the direct product of two copies of the Virasoro algebra. It is generated by $\left(L_n, \bar{L}_m\right)$ for $n, m \in \mathbb{Z}$. We will explore the Hilbert space of our CFT further below, but is a direct sum of irreducible unitary representations of the Virasoro algebra. Unitary will in this case mean that we will equip the Hilbert space with the conjugation $L_n^\dagger = L_{-n}$.

Physical interpretation. Recall that the global Virasoro generators have direct physical interpretation, important for the quantum theory. We impose that

- $L_0 + \bar{L}_0$ generates dilations. This is the Hamiltonian operator; it measures energy.
- $L_0 - \bar{L}_0$ generates rotations. This is the the angular momentum operator; it measures spin.
- L_{-1} and $\bar{L}_{-1}$ generate translations in space. They are the left and right momenta.
- L_1 and $\bar{L}_1$ generate the so-called special conformal transformations, which are related to inversions.

What does this mean for the generators? Well, we have some requirements on our Hamiltonian. We will requrire that both L_0 and $\bar{L}_0$ are simultaneously diagonalisable, and that $L_0 + \bar{L}_0$ is bounded from below. This makes sense for the future section of radial quantisation, as it suggests that the radial coordinate $|z|$

should play the role of time.

As we soon are about to see, the other Virasoro generators behave somewhat like the creation and annihilation operators familiar from the quantum harmonic oscillator. The L_{+n} are "lowering" operators, and we require that they all annihilate the vacuum state. Meanwhile, the L_{-n} are "raising" operators, and they create excitations with higher energy.

2.3 The Hilbert Space

The operators L_0 and $\bar{L}_0$ give a detailed understaning of the CFT Hilbert Space $\mathcal{H}_{\text{CFT}}$. They commute and are simultaneously diagonalisable, and describe a basis for the Hilbert space by their eigenvectors. Because of this, we have special names for their eigenvalues, showed in Tab. 1 below.

Operator	Eigenvalue	Terminology
$\left(L_0, \bar{L}_0\right)$	$(h, \bar{h})$	Conformal weights
$H = L_0 + \bar{L}_0$	$\Delta = h + \bar{h}$	Conformal dimension
$P = L_0 - \bar{L}_0$	$\bar{\Delta} = h - \bar{h}$	Angular momentum

Table 1: Table of operators, their eigenvalues and the terminology in terms of the (global) Virasoro algebra generators. There is also a *twist* for $2L_0$ but that will not be covered in this essay.

We are interested in irreducible representation of the Virasoro algebra. Let us therefore review the *highest-weight representation* of the global Virasoro algebra, by the standard method.

The Highest Weight Representation. We will assume unitary. Let $|\phi\rangle \in \mathcal{H}_{\text{CFT}}$ be an eigenvector of L_0 with weight h. Using the Virasoro algebra, we find that the state $L_n|\phi\rangle$ is also an L_0-eigenvector, with eigenvalue shifted by n :

$$L_0|\phi\rangle = h|\phi\rangle \quad \Longrightarrow \quad L_0\left(L_n|\phi\rangle\right) = \left(L_n L_0 - n L_n\right)|\phi\rangle = (h-n)\left(L_n|\phi\rangle\right)$$

What can we tell from this? Well, we can see that L_n lowers the eigenvalue (energy) of a state by n. Since the Hamiltonian is bounded from below, we except us to not lower the energy indefinitely. I.e, there must exist an eigenstate $|\psi\rangle$ to L_0- that is annihilated by L_n for all $n>0$. Such state $|\psi\rangle \in \mathcal{H}_{\text{CFT}}$ is called a *primary* or *highest-weight* state for which

$$L_0|\psi\rangle = h|\psi\rangle, \quad \bar{L}_0|\psi\rangle = \bar{h}|\psi\rangle, \quad \text{and} \quad L_n|\psi\rangle = \bar{L}_n|\psi\rangle = 0 \quad \text{for all } n>0.$$

Primaries are therefore states of *lowest* energy and conformal weight, in a given irreducible representation. Furthermore, we can see that its possible to excite a state using an arbitrary string of Virasoro generators L_{n_i} with $n_i < 0$. That is, we can increase the weight of a primary state $|\psi\rangle$. Using the Virasoro algebra to commute the generators past each other, we can always write a final increased state as a linear combination of other states of the form

$$L_{-n_1} \cdots L_{-n_k}|\psi\rangle, \quad 0 \leq n_1 \leq \cdots \leq n_k.$$

All of these states are by definition eigenvectors of L_0 with weights

$$h + \sum_{i=1}^{k} n_i = h + N,$$

where the integer N is called the *level*. States with level $N \geq 1$ are called *descendant*s, and the vector space generated by linear combinations of the descendants is called the *Verma module* $\mathcal{V}_h$.

Starting with each primary state $|\psi\rangle$, we can then build an infinite tower of descendants. The number of independent states at level N is $p(N)$, the number of integer partitions of N. See Fig. 2 for a Verma module

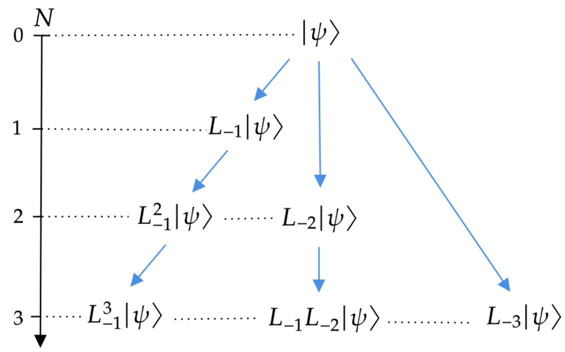


Figure 2: A tower of decendants to level $N = 3$ starting with a primary state $|\psi\rangle$.

to level 3.

Each Verma module $\mathcal{V}_h$ furnishes a representation of the Virasoro algebra, however, such a representation does not necessarily need to be irreducible. It may contain nontrivial subrepresentations and be reducible. We will approach this as the following. Eeach primary state $|\psi\rangle \in \mathcal{H}_{\text{CFT}}$, labeled by its weights $(h, \bar{h})$, generates a Verma module $\mathcal{V}_{h,\bar{h}} = \mathcal{V}_h \otimes \overline{\mathcal{V}}'_{\bar{h}}$ via the action of a string of Virasoro generators L_{-n_i}. If the Verma modules $\mathcal{V}_h$ or $\overline{\mathcal{V}}'_{\bar{h}}$ are reducible, they include null vectors (primaries that are also decendants). Finding them and quotient out the submodules of primaries they generate will leaves us with irreducible representations R_h and $\bar{R}'_{\bar{h}}$. In conclusion, the generated Hilbert space $\mathcal{H}_{\text{CFT}}$ is then a direct sum of these irreducible representations:

$$\mathcal{H}_{\text{CFT}} = \bigoplus_{h,\bar{h}} m_{h,\bar{h}} \left(R_h \otimes \bar{R}'_{\bar{h}} \right), \quad (h, \bar{h}) \in \mathcal{S},$$

where m is the mutliplicity and R_h and $\bar{R}'_{\bar{h}}$ are irreduceble. The *spectrum* of the CFT is the list $\mathcal{S}$ of the conformal weights of all primaries in the theory. There can be theories with either finitely or infinitely many primaries, but either way the minimal information needed to reconstruct the Hilbert space is in $\mathcal{S}$.

2.4 Classifications of CFTs

Equipped with the knowledge above, there are ways of classifying differt CFTs based on their representations. For instance, we have

- A CFT is *unitary* if $\mathcal{H}_{\text{CFT}}$ has a positive-definite inner product where $L_0 + \bar{L}_0$ is self-adjoint. For this, we require $L_n = L_{-n}^\dagger$. Only in unitary CFTs are all states normalisable, and only these CFTs are physical. Every unitary CFT has $c \geq 0$ and all weights $h, \bar{h} \geq 0$. The only state with $h = 0$ is the vacuum state $|0\rangle$. Same holds for c. Unitarity was assumed throughout above and will be assumed for all CFTs in this essay.
- A CFT is *compact* if its spectrum is real and discrete, and there if is a unique state with $\Delta = 0$. This state is the vacuum $|0\rangle$, and its associated field is the identity $\mathbb{1}$. The vacuum is $\text{PSL}(2, \mathbb{C})$-invariant, meaning that $L_n|0\rangle = 0$ for all $n \geq -1$.
- A CFT is *rational* if it has a finite number of primary states.

A few examples of 2d CFTS include the free boson and free fermion, both which are real, complex and compact.

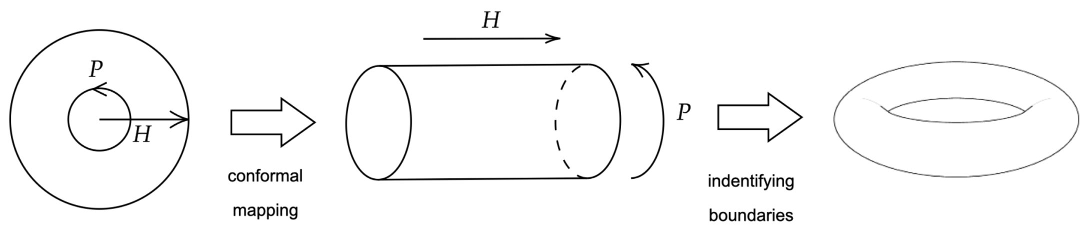


Figure 3: The process of arriving to to the torus.

2.5 Fields

By the operator-state correspondence, we allow the Virasoro algebra to act on fields as well as on states, and we distinguish between primary and descendant fields, which correspond to primary and descendant states, respectively. A primary field $\mathcal{O}_{h,\bar{h}}$ obeys

$$L_0\mathcal{O}_{h,\bar{h}} = h\mathcal{O}_{h,\bar{h}}, \quad \bar{L}_0\mathcal{O}_{h,\bar{h}} = \bar{h}\mathcal{O}_{h,\bar{h}}, \quad \text{and} \quad L_n\mathcal{O}_{h,\bar{h}} = \bar{L}_n\mathcal{O}_{h,\bar{h}} = 0 \quad \text{for all } n > 0.$$

Primary fields transform covariantly under any local conformal transformation $z \mapsto w(z)$, as follows:

$$\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \longmapsto \mathcal{O}_{h,\bar{h}}(w,\bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}}\mathcal{O}_{h,\bar{h}}(z,\bar{z}),$$

where it's now clear why h and $\bar{h}$ are called the conformal weights. They are somewhat a measure of how much the field conformally transforms. If the above is only satisfied for global conformal transformations, the fields are called quasi-primary, and their associated states are primary with respect to the global algebra PSL $(2, \mathbb{C})$.

In quantum field theory (QFT), fields are operator-valued distributions on spacetime, so it is often useful to think of the map $|\psi\rangle \longleftrightarrow \mathcal{O}_\psi(z,\bar{z})$ as an operator-state correspondence. Note that in any QFT with a vacuum state $|0\rangle$, any operator $\mathcal{O}_\psi$ in such theory *defines* a state by creating it from the vacuum: $|\psi\rangle \equiv \mathcal{O}_\psi(0)|0\rangle$. It is only in CFT that the map goes the other way too.

2.6 From the Plane to the Cylinder

We will now restate the radial quantisation of a 2d CFT seen in the lectures. The idea of radial quantization is that, after performing some algebraic manipulations, the Hamiltonian (which generates time translations) will correspond to the dilation operator. But as we have seen, this is precisely what $H = L_0 + \bar{L}_0$ does. Thus radial quantization is what motivates our earlier assumptions on the boundedness and self-adjointness of H, and explains why we wanted to treat scaling eigenvalues as energies. Our final goal is to end up with a theory on a torus, and we will now begin with the first step in this process (also illustrated in Fig. 3); the cylinder. A natural start of a theory is to consider it to live on the flat Minkowski plane: where time runs upward and the spatial direction is horizontal. However, as the theory is easier formulated in Euclidean signature; we will Wick rotate the metric and *compactify* the spatial direction. What we get is an Euclidean cylinder $\mathbb{R} \times S^1$. In other words, we exploit the conformal property of our plane to transform our theory to one on a cylinder.

Energy-momentum tensor. Recall* that under a transformation $z \mapsto f(z)$, the energy-momentum tensor $T(z)$ behaves as

$$T'_{\text{cyl}}(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12}S(f(z), z)$$

*For full derivation, see for instance [4, 1].

with the Schwarzian derivative defined as

$$S(w,z) = \frac{1}{(\partial_z w)^2}\left((\partial_z w)\left(\partial_z^3 w\right) - \frac{3}{2}\left(\partial_z^2 w\right)^2\right).$$

A cylinder with circumference L can be parameterised by the complex coordinate $\zeta = t + ix$, where $x \sim x + L$, and the conformal transformation $z = e^{2\pi\zeta/L}$ maps the cylinder onto the plane. As we want to go from the plane to the cylinder, we are interested in the inverse of this transformation;

$$z \mapsto \zeta = \frac{L}{2\pi}\log(z).$$

The energy-momentum tensor in these new coordinates is given by

$$T_{\text{cyl}}(\zeta) = \underbrace{\left(\frac{2\pi}{L}\right)^2 z^2}_{\left(\frac{d\zeta}{dz}\right)^2}\left(T(z) + \frac{c}{12}\underbrace{\overbrace{S(\log(z), z)}^{\text{Schwarz. Deriv.}}}_{-\frac{1}{2}z^{-2}}\right).$$

We then have

$$T_{\text{cyl}}(z) = \left(\frac{2\pi}{L}\right)^2\left(z^2 T(z) - \frac{c}{24}\right).$$

Assuming $\langle T(z)\rangle = \langle \bar{T}(\bar{z})\rangle = 0$, the Casimir energy can be computed to be

$$(L_{\text{cyl}})_0 = -\int_0^L \langle T_{00}\rangle = -\int_0^L \frac{\pi(c+\bar{c})}{12L^2} = -\frac{\pi(c+\bar{c})}{12L}, \tag{2}$$

where $\langle T_{00}\rangle = -\frac{1}{2\pi}\left(\langle T_{\text{cyl}}(\zeta)\rangle + \langle \bar{T}_{\text{cyl}}(\bar{\zeta})\rangle\right)$. This means that we have a shift in the energy zero mode;

$$(L_{\text{cyl}})_0 = L_0 - \frac{c}{24}.$$

Consequentely, the Hamiltonian and the angular momentum operator for the cylinder can now expressed as

$$\begin{aligned} H &= \frac{1}{2\pi}\int_{\omega_1}\left(\frac{2\pi}{L}\right)^2\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right), \\ P &= \frac{1}{2\pi}\int_{\omega_1}\left(\frac{2\pi}{L}\right)^2\left(L_0 + \bar{L}_0\right) = \frac{2\pi}{L}\left(L_0 + \bar{L}_0\right). \end{aligned} \tag{3}$$

This will be useful for the following sections.

2.7 From the Cylinder to the Torus

Up until now, we have been considering conformal field theories defined on either the complex plane or the Riemann sphere, which are associated with the tree-level in perturbation theory in string theory. However, in quantum field theories, one effective approach for determining the fields present in a theory is to analyse loop diagrams as all possible states can propagate in the loops. We will adopt this for conformal field theories, which means that we will need to study CFTs on Riemannian surfaces with higher genus. The one-loop diagram for CFTs corresponds to a torus, which will yield a consistency condition known as modular invariance. Modular invariance will turn out to be a crucial concept for understanding 2d CFTs.

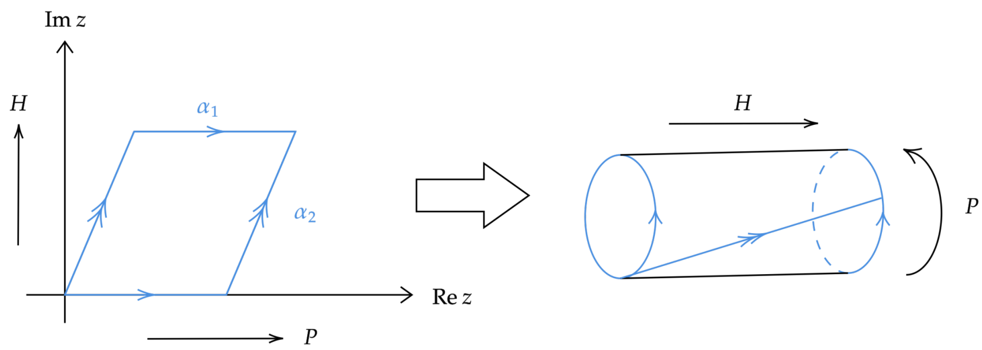


Figure 4: Illustration of getting from the plane to the cylinder, highlighting the directions of the Hamiltonian operator H and angular momentum operator P.

We will begin by considering our conformal field theory on a torus $\mathbb{T}^2$. We will arrive there by compactifying the cylinder, see Fig. 3. This is obtained by cutting out a finite piece from the infinite cylinder and identifying the ends, so both the space coordinate and time coordinate become periodic. However, before gluing the ends together, there is the possibility to twist the ends of the cylinder and it is therefore natural to parameterise the torus on a lattice. The key in this process is the complex *modular parameter* τ allowing us to identify points z in the complex plane as

$$z \sim z + m\alpha_1 + n\alpha_2, \quad m, n \in \mathbb{Z}$$

where (α_1, α_2) is a pair of complex numbers. As illustrated in the left schematic in Fig. **??**, this pair spans a lattice, whose smallest cell is called the *fundamental domain* of the torus. From a geometrical point of view, the torus is then obtained by identifying opposite edges of the fundamental domain. The modular parameter describing the shape of the torus is then defined as

$$\tau = \frac{\alpha_2}{\alpha_1} = \tau_1 + i\tau_2.$$

However, there are different choices of (α_1, α_2) giving the same lattice, and thus the same torus. This is something we need analyse further. Let us therefore assume that (α_1, α_2) and (β_1, β_2) both describe the same lattice. See Fig. 5 for an example of this. Now, this means that we can write the pair (β_1, β_2) as

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}.$$

Of course we can also express (α_1, α_2) in terms of (β_1, β_2), which amounts to the computation of the inverse relation

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

In general, for the inverse matrix to also have integer entries, we have to require that $ad - bc = \pm 1$ which just means that the unit cells in each basis should have the same volume (up to a sign). Moreover, we can span the same lattice by (α_1, α_2) and its negative $(-\alpha_1, -\alpha_2)$, so we can divide out a $\mathbb{Z}_2$ action. As expected, matrices with these properties are exactly the elements of the modular group already mentioned. For convenience, we will often choose $(\alpha_1, \alpha_2) = (1, \tau)$ so the identifications are now

$$z \sim z + 1 \sim z + \tau,$$

and in particular that the modular group of the torus now act as on the modular parameter as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}).$$

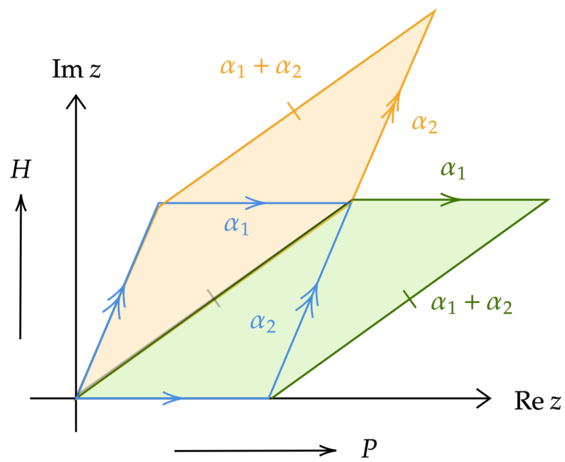


Figure 5: Example of a lattice equally described by (α_1, α_2) and by the two pairs $(\alpha_1 + \alpha_2, \alpha_2)$ and $(\alpha_1, \alpha_1 + \alpha_2)$.

In particular, we define the specific $\mathcal{T}$- and $\mathcal{S}$-transformation as

$$\begin{aligned} \mathcal{T}: &\quad \tau \mapsto \tau + 1, \\ \mathcal{S}: &\quad \tau \mapsto -\frac{1}{\tau}, \end{aligned}$$

where $\mathcal{S}^2 = \mathbb{1}$ and $(\mathcal{S}\mathcal{T})^3 = \mathbb{1}$. The elements $\mathcal{S}$ and $\mathcal{T}$ are the generators of the modular group, meaning that invariance under these transformations is sufficient to show invariance under the modular group. Note that because of the modular group, τ will be kept in the upper-plane.

2.8 The Torus Partition Function

Modular transformations of τ do not change the theory on the torus, so when investigating conformal field theories on the torus, we will need to *require* that partition functions are invariant under modular transformations. In this section we will find the partition function of the torus, and in the next we will explore the constraints modular invariance put on our theory.

Let us recall the partition function

$$Z = \operatorname{Tr} e^{-\beta H}, \tag{4}$$

for a statistical system with β as the inverse temperature. To find something analogous for our conformal field theory, we will work mainly in the operator formalism. First, we need to specify space and time directions on the torus. We will do this accordingly to what is illustrated in Fig. 3 and Fig. 4 and take space to run along the real axis and time along the imaginary axis. Let H be the Hamiltonian generating translations along the time direction and P the total momentum generating translations along the space direction of the torus, given by the expression derived from the cylinder in Eq. (3). Then the operator which translates the system parallel to the period ω_2 over a distance a in Euclidean space-time is given by

$$\exp\left(-\frac{a}{|\omega_2|}\left[H \operatorname{Im}\omega_2 - iP \operatorname{Re}\omega_2\right]\right).$$

If we regard a as a lattice spacing, then this operator will take us from one row of a lattice to the next. If the complete period contains m lattice spacings, that is $|\omega_2| = ma$, then we will obtain the partition function by taking the trace of the translation operator to the m-th power:

$$Z(\omega_1, \omega_2) = \operatorname{Tr} \exp\left(-H \operatorname{Im} \omega_2 + iP \operatorname{Re} \omega_2\right), \tag{5}$$

which is a generalisation of Eq. (4) and the appearens of P arises from the possibility of twisting the ends of the cylinder when forming the torus.

We want to relate this partition function to the modular parameter $\tau = \tau_1 + i\tau_2$. In order to do so, we will define $\omega_1 = L$, the circumference of the cylinder. This means that we have

$$\begin{aligned} \tau_1 &= \frac{\operatorname{Re}\omega_2}{\omega_1} = \frac{\operatorname{Re}\omega_2}{L}, \\ \tau_2 &= \frac{\operatorname{Im}\omega_2}{\omega_1} = \frac{\operatorname{Im}\omega_2}{L}. \end{aligned}$$

Using our expressions for H and P from the cylinder in Eq. (3), we can now rewrite the full partition function of the torus in (5), in terms of the modular parameter, as

$$\begin{aligned} Z(\tau, \bar{\tau}) &= \operatorname{Tr} \exp \pi i \left\{ (\tau - \bar{\tau}) \left(L_0 + \bar{L}_0 - c/12 \right) + (\tau + \bar{\tau}) \left(L_0 - \bar{L}_0 \right) \right\} \\ &= \operatorname{Tr} \exp 2\pi i \left\{ \tau \left(L_0 - c/24 \right) - \bar{\tau} \left(\bar{L}_0 - c/24 \right) \right\}. \end{aligned}$$

We will now define the parameters

$$q = \exp 2\pi i \tau, \quad \bar{q} = \exp -2\pi i \bar{\tau},$$

to arrive to our final expression for the torus partition function

$$\text{Torus Partition Function:} \quad \boxed{Z(\tau, \bar{\tau}) = \operatorname{Tr}\left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \right)} \tag{6}$$

As we have assumed unitary, the trace is over the physical states in the Hilbert space and thus does not include null-states.

Remark. The partition function only depends on two key properties of the torus CFT: the central charge and the modular parameter τ (that is, on the shape of the torus/the twisted gluing condition). No dependence on the *size* of the torus exists, which is consistent with the conformal property of CFTs.

2.8.1 Modular invariance

Note that since $\mathrm{PSL}(2, \mathbb{C})$ transformations of the modular parameter τ do not change the torus, the CFT, and in particular the partition function (6) has to be invariant under the action of the modular group. It is the main goal of this chapter to study this question which imposes strong constraints on the combination of chiral and anti-chiral fields. In order to get accustomed to this concept, we will introduce a way of decomposing the Hilbert space to express the partition function differently.

Virasoro characters. To a Verma module generated by the Virasoro generators L_{-n} $(n > 0)$ acting on the highest-weight state $|h\rangle$, we associate a generating function $\mathcal{X}_{(h)}$ as

$$\mathcal{X}_{(h)}(\tau) = \operatorname{Tr}_{\text{descendants of } (h)} q^{L_0 - c/24}$$

where as usual $q = e^{2\pi i \tau}$, τ is the complex modular parameter and the factor of $q^{-c/24}$ is motivated by the modular invariance, as we will soon see. The trace is over all states contained in the Verma module associated with N. The expression for the non-holomorphic character $\bar{\chi}_{(\bar{h})}(\bar{\tau})$ is analogous. The characters are generating functions for the degeneracy at level N. A way of seeing this is to simply count how many linearly independent states descend at level n from our given primary. The general expression is given by $\dim(h+n)$ and recalling the discussion above, a given state has weight $h+N$. We can thus rewrite the characters as

$$\mathcal{X}_{(h)}(\tau) = \sum_{N=0}^{\infty} \dim(h+N) q^{N+h-c/24} = q^{h-c/24} \sum_{N=0}^{\infty} \dim(h+N) q^N.$$

The number of (possibly dependent) states at level N is given by the partition function $p(N)$, the number of partitions of the integer N. We therefore have that $\dim(h+N) \leq p(N)$. This implies that for the series to be uniformly covergent, we require that $|q| < 1$ i.e., for τ in the upper half-plane. This agrees with our discussion of the action of the modular group of the modular parameter. For fields that have no null descendants we can then define the generic Virasoro character using the well-known generating function for number theoretical partition functions as

$$\mathcal{X}_{(h)}(\tau) = q^{h-c/24} \sum_{N=0}^{\infty} p(N) q^N = q^{h-c/24} \prod_{N=0}^{\infty} \frac{1}{1-q^N}.$$

It will be useful for us to define the *Dedekind Eta* function:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right).$$

The Dedekind Eta function is modular invariant, however it has an even stronger property of also being a *modular form*:

Definition 2.2. Modular form. A modular form $f_k(\tau)$ of weight k is a holomorphic function of the complex variable τ valued in the upper half-plane, satisfying the following properties:

1. $f_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f_k(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}),$
2. f is required to be bounded as $\tau \to i\infty$ $(q \to 0)$.

The Dedekind Eta function is a modular form of weight 1/2, and it transforms under $\mathcal{S}$ and $\mathcal{T}$ as

$$\boxed{\begin{aligned} \mathcal{T}: \quad \eta(\tau+1) &= e^{i\pi/12}\eta(\tau) \\ \mathcal{S}: \quad \eta(-1/\tau) &= \sqrt{-i\tau}\,\eta(\tau) \end{aligned}} \tag{7}$$

See Appendix 5 for proof of this.

We now note that the characters are, in general, proportional to $1/\eta(\tau)$. In particular, the generic Virasoro character can now be written like

$$\chi_{(h)}(\tau) = \frac{q^{h+(1-c)/24}}{\eta(\tau)}. \tag{8}$$

Partition function decomposition. The purpose of introducing these characters is to find conditions on our partition function imposed by modular invariance. As we saw above, we can decompose the Hilbert space into a direct sum of tensor products of irreducible representations, we can rewrite the partition function as a sum over products of characters of the holomorphic and non-holomorphic representations as

$$Z(\tau,\bar{\tau}) = \sum_{i,j} M_{ij} \mathcal{X}_i(\tau) \mathcal{X}_j(\bar{\tau}). \tag{9}$$

Here i and j label the highest weight state $|i,j\rangle$ where i and j are used for representations of the holomorphic and anti-holomorphic algebra, respectively. The multiplicity of such a state is M_{ij}, a non-negative integer. The existence and uniqueness of the vacuum requires $M_{00}=1$ however the other elements are strongly constrained by our requirement of modular invariance. These steps can be familiar from group theory, where a character of a group element g is a trace over some representation $\chi(g)_R$ of g. In Lie algebra language, we are taking the traces of Casimir elements.

We know that our partition function needs to be modular invariant, how do the characters transform according to the modular transformations $\mathcal{S}$ and $\mathcal{T}$ of the modular group?

$\mathcal{T}$-transformation. Recall that $\mathcal{T}$ transforms τ to $\tau+1$. Plugging this in for a Virasoro character yields

$$\chi_a(\tau+1) = e^{2\pi i\left(h_a - \frac{c}{24}\right)}\chi_a(\tau),$$

and we thus have

$$\chi_a(\tau+1) = \sum_b T_{ab}\chi_b(\tau),$$

where $T_{ab} = \delta_{ab}e^{2\pi i\left(h_a-\frac{c}{24}\right)}$.

$\mathcal{S}$-transformation. This transformation is a bit trickier to show.

$$\chi_i\left(-\frac{1}{\tau}\right) = \sum_j S_{ij}\chi_j(\tau),$$

where S is a symmetric and unitary matrix. Using the unitarity of S and T the modular invariance, the conditions on our matrices are

$$[M,T]=0 \quad \text{and} \quad [M,S]=0.$$

These conditions are trivially satisfied for $M_{ij}=\delta_{ij}$. Such a theory is called a *diagonal* CFT. One example is the Ising model.

2.8.2 Example: The Free Boson

Let's review an example of a modular invariant partition function. In this section we will find the partition function of the free boson on a torous. This CFT has central charge $c=1$ and the complete free boson Hilbert space is obtained by acting on states $|k\rangle \equiv e^{ik\cdot\hat{x}}|0\rangle$ with all the left- and right-moving creation operators;

$$(L_{-l})^{i_l}\dots(L_{-1})^{i_1}\left(\bar{L}_{-m}\right)^{j_m}\dots(L_{-1})^{j_1}|k\rangle,$$

where unitary implies positive norms and the standard Hermitian conjugation rules $L_n^\dagger = L_{-n}, \bar{L}_n^\dagger = \bar{L}_{-n}$, the oscillator commutation relations, and the norm $\langle k \mid k\rangle = 1$ for the momentum states. Here $\hat{x}$ is the zero-mode canonically conjugate to the 'momentum'* $\hat{p}\equiv L_0 = \bar{L}_0$. The non-zero mode part can be easily be found. For each L_n we get a partition function of a boson $(1-q^n)^{-1}$, and we simply just need to multiply them. Given the above, we then expect the partition function of the free boson to behave as

$$Z_{\text{boson}}(\tau,\bar{\tau}) \propto \frac{1}{\eta(\tau)}\frac{1}{\eta(-\bar{\tau})}Z_{0-\text{mode}} = \frac{1}{|\eta(\tau)|^2}Z_{0-\text{mode}}.$$

The full Hilbert space is a product of the independent contributions from the different oscillators L_{-n} and the 'momentum' k. We will now however disregard the zero-mode from the partition function†, which means

*'Momentum' here is not 2d momentum, but rather a global internal symmetry of the 2d CFT, which in string theory is interpreted as space-time momentum.

†Using path-integral formalism, one can show the transformation properties explicitly for the full partition function, see for instance [4].

that the trace will be evaluated over the Fock space associatied with the vacuum, the identity operator. The term $1/|\eta(\tau)|^2$ is not modular invariant, so we need to impose a proportional constant that will keep the modular invariance. We will impose that the constant is given by

$$Z_{\text{boson}}(\tau,\bar{\tau}) = \frac{1}{\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2},$$

and that this partition function is modular invariant.

Proof. We will use the properties of how the Dedekind function transforms under $\mathcal{S}$ and $\mathcal{T}$ from Eq. (7). The invariance under $\mathcal{T}$ is trivial because η just picks up a phase, and $\operatorname{Im}(\tau+1) = \operatorname{Im}\tau$:

$$Z_{\text{boson}}(\tau+1,\bar{\tau}+1) = \frac{1}{\sqrt{\operatorname{Im}(\tau+1)}\left|\eta\left(\tau+1\right)\right|^2} = \frac{1}{\sqrt{\operatorname{Im}\tau}\left|\eta(\tau)\right|^2|e^{i\pi/12}|^2} = \frac{1}{\sqrt{\operatorname{Im}\tau}\left|\eta(\tau)\right|^2} = Z_{\text{boson}}(\tau,\bar{\tau}).$$

Under $\mathcal{S}$, we first note that we have $-1/\tau = -\bar{\tau}/(\tau\bar{\tau})$ so that $\operatorname{Im}\left(-1/\tau\right) = \operatorname{Im}\tau/|\tau|^2$. The rest is then simple:

$$\begin{aligned}
Z_{\text{boson}}(-1/\tau,-1/\bar{\tau}) &= \frac{1}{\sqrt{\operatorname{Im}\left(-\frac{1}{\tau}\right)}\left|\eta\left(-\frac{1}{\tau}\right)\right|^2} \\
&= \left(\sqrt{\frac{\operatorname{Im}\tau}{|\tau|^2}}\left|\eta\left(-\frac{1}{\tau}\right)\right|^2\right)^{-1} \\
&= \left(\sqrt{\frac{\operatorname{Im}\tau}{|\tau|^2}}\sqrt{|\tau|^2}|\eta(\tau)|^2\right)^{-1} \\
&= \frac{1}{\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2} = Z_{\text{boson}}(\tau,\bar{\tau}).
\end{aligned}$$

□

A Modular invariance of Dedekind Eta function

In this section, we will show that the Dedekind Eta function $\eta(\tau)$ is modular invariant under the $\mathcal{T}$ and $\mathcal{S}$ transformations. The non-holomorphic proofs are obmitted.

A.1 $\mathcal{T}$-transformation

$$\boxed{\mathcal{T}: \quad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)}$$

Proof. Under the modular $\mathcal{T}$-transformation $\mathcal{T}: q \mapsto e^{2\pi i}q$, the Dedekind Eta function transforms as

$$\eta(\tau+1) = e^{\frac{2\pi i}{24}} q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n} q^n\right) = e^{\frac{2\pi i}{24}} \eta(\tau) = e^{i\pi/12}\eta(\tau).$$

□

A.2 $\mathcal{S}$-transformation

$$\boxed{\mathcal{S}: \quad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)}$$

Proof. The invariance under the $\mathcal{S}$-transformation is a bit harder to show. For this we need the *Poisson resummation formula* relating sums of exponentials, which will heuristically (indirectly) be derived here. Suppose we have the sum

$$Z(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2}$$

Suppose the origin of the term n^2 in the exponent came from a Gaussian integral

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2 + i\beta x} = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\beta^2}{4\alpha}},$$

such that

$$4\alpha = -\frac{1}{i\pi\tau}, \quad \alpha = \frac{i}{4\pi\tau}.$$

We can then evaluate the integral as

$$\int_{-\infty}^{\infty} dx e^{-\frac{i}{4\pi\tau}x^2 + inx} = \sqrt{-4i\pi^2\tau} e^{i\pi\tau n^2}.$$

The partition function can then be described as

$$Z(\tau) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{-4i\pi^2\tau}} e^{-\frac{i}{4\pi\tau}x^2} e^{inx}.$$

Let us first consider the sum-term involving e^{inx}. For this we will make use of the discrete Fourier transform of the periodic function δ

$$\sum_{n=-\infty}^{\infty} e^{inx} = \sum_{m=-\infty}^{\infty} 2\pi\delta(x - 2\pi m).$$

In any interval like $0 \leq x < 2\pi$, we have

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi\delta(x),$$

where the normalisation follows upon integrating both sides in x. The left-hand side of this equation contributes only with 2π from the term $n = 0$, all other terms are zero. Thus we have

$$Z(\tau) = \int_{-\infty}^{\infty} dx \sum_{m=-\infty}^{\infty} 2\pi\delta(x - 2\pi m) e^{-\frac{i}{4\pi\tau}x^2}$$

Computing the integral sets sets $x = 2\pi m$. So we finally have have

$$Z(\tau) = \sum_{m=-\infty}^{\infty} \frac{2\pi}{\sqrt{-4i\pi^2\tau}} e^{-\frac{i}{2\pi\tau}4\pi^2 m^2} = \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{-i\tau}} e^{-\frac{i}{\tau}\pi m^2}.$$

That is, we have shown that

$$Z(\tau) = \frac{1}{\sqrt{-i\tau}} Z\left(-\frac{1}{\tau}\right).$$

□

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